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The anharmonic oscillator: perturbation series for cubic and quartic energy distortion

J E Drummond

Department of Applied Mathematics, Science Faculty, Australian National University, Canberra, Australia

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Abstract. Twenty five terms of the perturbation series are calculated for the first five energy levels of an oscillator with λx^4 energy distortion and a similar calculation is carried out for the first six levels for λx^3 distortion. For the positive quartic the alternating series is summed using the Aitken δ_2 transformation for λ from 0.1 to 100. An exponential integral model and a truncated binomial model are used to transform the series for negative distortion energies, and the complex energy eigenvalues are calculated for $(-\lambda)$ from 0.01 to 1.

1. Introduction

The harmonic oscillator in classical and quantum mechanics is a rare phenomenon, but it is a basic first approximation to a wide variety of small-amplitude anharmonic oscillating systems. If the potential energy of an oscillator can be expanded as a power series in x, where x is the displacement from the minimum position, then the potential energy has the form

$$V = V_0 + Ax^2 + Bx^3 + Cx^4 + \ldots .$$

The constant A is positive for a harmonic oscillator. The next most important term is the cubic term for an asymmetric oscillator or the quartic term for a symmetric oscillator. Hence these two terms are worthy of study. So far the positive perturbations have been studied in some detail, but the negative perturbations have received somewhat less attention.

The energy levels of the Schrödinger wave equation,

$$(-\mathrm{d}^2/\mathrm{d}x^2 + x^2 + \lambda x^m)\psi = E\psi, \tag{1}$$

have been calculated by Biswas *et al* (1973) for eight energy levels for $m = 4$ and two levels for $m = 6$ and 8, for λ from 0.1 to 100. Their method was to find the zeros of determinants which converge to the Hill determinant. Ginsberg and Montroll (1978) have shown how to use good approximate wavefunctions to calculate the energy for all positive λ. Bazley and Fox (1961) have also found rigorous upper and lower bounds for the lower energy levels for $m = 4$.

Bender and Wu (1969) have calculated 75 terms of the ground-state energy perturbation series for the positive quartic energy perturbation and have demonstrated that the ratio of successive terms increases linearly. This means that the series is similar

to the asymptotic series for the exponential integral and that the radius of convergence of their series is zero. However, Simon (1970) has studied the analytic properties of this series and its Padé approximants (Padé 1892), while Graffi *et al* (1970) have used both Padé approximants on 40 terms and Padé–Borel approximants on 20 terms for λ (their β) from 0.1 to 14, to calculate energies. These agree with the results of Biswas *et al*.

The following calculations show that convergence speeding of the asymptotic series for positive quartic distortion gives an unbiased sum up to at least $\lambda = 50$. On the other hand, rounding-off errors starting with 16-figure accuracy cause the accuracy to decrease from 14 figures at $\lambda = 0.1$ to two figures at $\lambda = 50$.

Calculations of the negative quartic and cubic distortions are also made, using the direct perturbation series together with an improved truncation based on the exponential integral. An even better method is a truncated binomial transformation (Drummond 1981), which has been successfully used to find exponential integrals in the shadow of their singularity and also to find complex energy levels in the Stark effect.

2. Perturbation series for quartic and cubic distortion

We wish to find the eigenvalues $E(\lambda)$ of the differential equation

$$(-\mathrm{d}^2/\mathrm{d}x^2 + x^2 + \lambda x^m)\psi_N(\lambda, x) = E_N(\lambda)\psi_N(\lambda, x) \qquad (m = 3, 4; N = 0, 1, 2, \ldots)$$

with the associated boundary condition

$$\lim_{x \to \pm\infty} \psi_N(\lambda, x) = 0.$$

To do this we set

$$\psi_N(\lambda, x) = \exp(\tfrac{1}{2}x^2)[\psi_{N,0}(x) + \lambda\psi_{N,1}(x) + \lambda^2\psi_{N,2}(x) + \ldots]$$

and

$$E_N(\lambda) = 2N + 1 + \lambda E_{N,1} - \lambda^2 E_{N,2} + \lambda^3 E_{N,3} - \ldots. \tag{2}$$

The first 25 coefficients of equation (2) for the quartic distortion for five energy levels and 20 coefficients for the cubic for six levels were computed, extending the method of Bender and Wu (1969, p 1233) using 8- and 16-figure accuracy. These two calculations agree mostly to six figures, so the higher-precision calculation was truncated at 14 figures, except for the coefficients for the last two cubic energy levels. These agreed to four figures, and so were truncated at 12 figures.

For the quartic energy distortion oscillator, the values of $E_{N,r}$ are given in table 1, while for the cubic energy distorted oscillator, the odd coefficients are all zero and the values of $E_{N,2r}$ are given in table 2.

3. Energy levels for quartic distortion and positive λ

Graffi *et al* (1970) calculated $E_0(\lambda)$ using the series of Bender and Wu (1969). They found that the Padé–Borel transformation was more rapidly convergent than the Padé transformation, and noted that the diagonal Padé approximants tend to a constant for large λ whereas $E_0(\lambda) \sim \lambda^{1/3}$ for large λ.

Table 1. Values of the unperturbed energy level and the next 25 coefficients, $E_{N,r}$ in the perturbation series (2) for the first five energy levels of the oscillator with quartic energy distortion. The figure after the comma is the power of 10 multiplying the number.

r	$E_{0,r}(E_{0,0}=1)$	$E_{1,r}(E_{1,0}=3)$	$E_{2,r}(E_{2,0}=5)$	$E_{3,r}(E_{3,0}=7)$	$E_{4,r}(E_{4,0}=9)$
1	0.75 , 0	3.75 , 0	9.75 , 0	1.875 , 1	3.075 , 1
2	1.312 5 , 0	1.031 25 , 1	3.843 75 , 1	9.843 75 , 1	2.030 625 , 2
3	5.203 125 , 0	6.117 187 5 , 1	3.137 343 75 , 2	1.044 140 625 , 3	2.674 265 625 , 3
4	3.016 113 281 25 , 1	5.082 861 328 125 , 2	3.492 436 523 437 5 , 3	1.482 868 652 343 7 , 4	4.662 619 628 906 2 , 4
5	2.238 112 792 968 7 , 2	5.201 290 283 203 1 , 3	4.687 465 649 414 1 , 4	2.502 386 169 433 6 , 5	9.566 799 587 402 3 , 5
6	1.999 462 921 142 6 , 3	6.185 748 733 520 5 , 4	7.175 835 264 587 4 , 5	4.757 638 423 919 7 , 6	2.192 983 547 085 6 , 7
7	2.077 708 948 516 8 , 4	9.285 768 958 663 9 , 5	1.215 698 315 810 4 , 7	9.900 837 385 654 4 , 7	5.461 106 326 994 8 , 8
8	2.456 891 772 873 4 , 5	1.226 637 891 862 0 , 7	2.238 102 217 455 4 , 8	2.216 059 059 135 0 , 9	1.452 593 665 772 3 ,10
9	3.256 021 887 746 8 , 6	1.982 444 038 290 7 , 8	4.425 055 676 127 1 , 9	5.274 196 085 772 3 ,10	4.081 491 036 275 3 ,11
10	4.781 043 106 012 5 , 7	3.468 183 104 375 6 , 9	9.321 427 957 285 1 ,10	1.324 361 967 898 7 ,12	1.202 259 776 809 1 ,13
11	7.708 333 164 092 8 , 8	6.527 398 391 289 0 ,10	2.080 423 056 168 8 ,12	3.489 345 053 740 2 ,13	3.692 555 289 483 1 ,14
12	1.354 432 468 922 9 ,10	1.315 416 766 383 0 ,12	4.899 902 515 353 3 ,13	9.608 371 218 796 0 ,14	1.177 841 929 154 2 ,16
13	2.577 262 349 393 4 ,11	2.827 603 549 273 7 ,13	1.214 244 899 767 0 ,15	2.757 222 141 234 4 ,16	3.890 515 423 213 6 ,17
14	5.281 751 322 678 4 ,12	6.462 619 350 426 3 ,14	3.158 870 905 492 3 ,16	8.227 892 397 182 3 ,17	1.327 821 926 033 9 ,19
15	1.160 166 746 583 1 ,14	1.566 045 280 266 3 ,16	8.611 792 472 722 0 ,17	2.549 280 860 067 6 ,19	4.674 959 465 640 5 ,20
16	2.719 757 615 246 9 ,15	4.013 194 793 106 4 ,17	2.456 717 688 091 5 ,19	8.191 217 146 224 8 ,20	1.695 881 007 658 3 ,22
17	6.778 794 692 977 2 ,16	1.085 007 847 182 1 ,19	7.324 289 197 199 9 ,20	2.727 048 186 775 0 ,22	6.332 930 558 404 1 ,23
18	1.790 210 195 015 5 ,18	3.087 899 189 955 9 ,20	2.279 411 507 700 6 ,22	9.400 424 342 333 8 ,23	2.432 894 384 108 4 ,25
19	4.994 011 921 119 7 ,19	9.231 386 580 424 6 ,21	7.396 946 044 233 1 ,23	3.353 239 265 793 4 ,25	9.610 529 104 855 5 ,26
20	1.467 514 010 204 4 ,21	2.893 216 072 582 8 ,23	2.500 303 815 001 3 ,25	1.237 169 622 174 6 ,27	3.902 393 462 072 0 ,28
21	4.531 136 296 684 8 ,22	9.488 325 779 811 6 ,24	8.794 008 610 707 4 ,26	4.718 921 791 943 9 ,28	1.628 430 471 253 3 ,30
22	1.466 652 370 037 3 ,24	3.250 307 132 935 2 ,26	3.214 991 286 241 6 ,28	1.849 987 566 907 8 ,30	6.982 043 824 343 6 ,31
23	4.966 283 069 462 7 ,25	1.161 079 009 509 8 ,28	1.220 445 451 374 4 ,30	7.572 360 438 408 9 ,31	3.075 431 760 213 5 ,33
24	1.755 839 492 534 9 ,27	4.318 387 635 070 9 ,29	4.805 658 842 288 6 ,31	3.182 710 024 931 3 ,33	1.391 487 802 899 7 ,35
25	6.470 221 042 946 6 ,28	1.669 803 282 377 0 ,31	1.960 829 257 723 2 ,33	1.380 339 902 233 1 ,35	6.466 052 892 688 0 ,36

Table 2. Values of the unperturbed energy level, $E_{N,0}$, and some non-zero coefficients, $E_{N,r}$, in the perturbation series (2) for the first six energy levels of the oscillator with cubic energy distortion. The figure after the comma is the power of 10 multiplying the number.

r	$E_{0,r}(E_{0,1}=1)$	$E_{1,r}(E_{1,0}=3)$
2	0.687 5 , 0	4.437 5 , 0
4	1.816 406 25 , 0	2.197 265 625 , 1
6	9.694 580 078 125 , 0	2.020 358 886 718 7 , 2
8	7.343 599 319 458 0 , 1	2.476 413 288 116 5 , 3
10	7.020 215 699 672 7 , 2	3.636 194 486 832 6 , 4
12	8.009 464 054 293 9 , 3	6.085 723 494 591 2 , 5
14	1.056 594 310 432 2 , 5	1.129 556 215 750 4 , 7
16	1.579 388 457 500 6 , 6	2.286 559 099 151 1 , 8
18	2.637 680 143 864 3 , 7	4.993 837 044 556 5 , 9
20	4.869 803 661 706 8 , 8	1.168 094 227 291 0 ,11
22	9.855 972 701 400 7 , 9	2.911 227 314 657 8 ,12
24	2.171 426 811 198 4 ,11	7.701 990 056 471 9 ,13
26	5.176 557 787 439 4 ,12	2.156 889 659 809 4 ,15
28	1.328 329 756 386 8 ,14	6.379 308 575 789 3 ,16
30	3.651 898 312 453 0 ,15	1.988 928 399 188 2 ,18
32	1.071 210 475 479 7 ,17	6.525 826 623 762 3 ,19
34	3.340 063 525 638 9 ,18	2.249 798 812 985 8 ,21
36	1.103 314 683 780 5 ,20	8.137 468 639 313 4 ,22
38	3.849 375 115 858 1 ,21	3.083 400 259 536 0 ,24
40	1.414 594 475 405 9 ,23	1.222 164 653 717 2 ,26

r	$E_{2,r}(E_{2,0}=5)$	$E_{3,r}(E_{3,0}=7)$
2	1.193 75 , 1	2.318 75 , 1
4	9.169 921 875 , 1	2.440 429 687 5 , 2
6	1.264 970 458 984 4 , 3	4.555 002 197 265 6 , 3
8	2.261 945 989 608 8 , 4	1.089 017 999 839 8 , 5
10	4.729 827 285 168 2 , 5	3.010 991 602 530 2 , 6
12	1.103 352 806 530 7 , 7	9.190 755 084 886 6 , 7
14	2.798 388 318 737 5 , 8	3.020 045 801 329 8 , 9
16	7.596 734 338 938 6 , 9	1.052 129 304 844 7 ,11
18	2.184 958 349 030 8 ,11	3.847 703 743 063 8 ,12
20	6.612 100 109 743 4 ,12	1.467 077 245 302 6 ,14
22	2.095 074 294 403 6 ,14	5.803 930 930 424 7 ,15

r	$E_{4,r}(E_{4,0}=9)$	$E_{5,r}(E_{5,0}=11)$
2	3.818 75 , 1	5.693 75 , 1
4	5.120 507 812 5 , 2	9.287 695 312 5 , 2
6	1.210 688 696 29 , 4	2.663 363 256 84 , 4
8	3.645 814 246 56 , 5	9.697 185 987 28 , 5
10	1.262 559 280 17 , 7	4.047 624 363 34 , 7
12	4.800 718 321 72 , 8	1.849 278 485 06 , 9
14	1.954 666 212 50 ,10	9.019 410 557 12 ,10
16	8.394 116 319 60 ,11	4.625 551 168 03 ,12
18	3.764 737 241 45 ,13	2.469 954 146 60 ,14
20	1.751 521 300 30 ,15	1.364 027 460 75 ,16
22	8.412 434 314 36 ,16	7.753 006 489 04 ,17

This last comment is not surprising, since the diagonal Padé approximants $[N, N]$ are the ratio of two polynomials of equal degree and are only expected to imitate series for small to moderate sized λ, in this case for λ up to 100 as in table 3.

Table 3. Values of $E_N(\lambda)$ calculated from the perturbation series using repeated Aitken transformations. The digits in brackets are the earliest digits to disagree with the values calculated by Biswas *et al.*

λ	$E_0(\lambda)$	$E_1(\lambda)$	$E_2(\lambda)$	$E_3(\lambda)$	$E_4(\lambda)$
0.1	1.065 285 509 54(6)	3.306 872 013(4)	5.747 959 27(0)	8.352 677 8(4)	11.098 595(7)
0.2	1.118 292 654 3(7)	3.539 005 3(8)	6.277 248(8)	9.257 766(2)	12.440 60(3)
0.3	1.164 047 1(8)	3.732 484(5)	6.705 72(0)	9.975 31(6)	13.488 88(8)
0.4	1.204 810 3(9)	3.901 087(8)	7.072 598(6)	10.582 5(6)	14.368 9(3)
0.5	1.241 854(2)	4.051 93(4)	7.396 9(1)	11.115 1(7)	15.136 8(8)
0.6	1.275 983(9)	4.189 28(7)	7.689 5(8)	11.593 1(7)	15.823 5(6)
0.7	1.307 749(3)	4.315 94(8)	7.957 5(8)	12.029 0(6)	16.488(0)
0.8	1.337 54(6)	4.433 8(6)	8.205 7(0)	12.431 2(4)	17.022(9)
0.9	1.365 67(1)	4.544 4(6)	8.437 3(5)	12.805(7)	17.557(3)
1	1.392 35(5)	4.648 8(2)	8.655 0(0)	13.156(9)	18.057(7)
10	2.449(4)	8.6(0)	16.6(5)	25.8(4)	35.9(3)
20	2.9(9)	10.(9)	20.6(5)	31.(8)	44.(6)
50	4.0(3)	13.(8)	2(6).	3(8)	5(9)
100	5.0(1)	1(7)	2(7).	4(5)	7(1)

Shanks (1955) observed that repeated use of his e_1 transformation, also called the Aitken (1925–6) δ_2 and Padé [1, 2] (Padé 1892), was better than using higher-order transformations, so this was tried on 25 terms of the asymptotic series for $E_0(\lambda)$ to $E_4(\lambda)$ for λ from 0.1 to 100.

The results are given in table 3.

The successive Aitken transformations converged for eight steps, using up 16 terms, and then became erratic due to rounding-off errors. They are truncated before the first obviously erratic term, which was also used to estimate the rounding-off error. This was confirmed by comparison with the table of Biswas *et al* (1973). One inaccurate digit is included in brackets in table 3.

The energies given in table 3 using the eight Aitken transformations on 16 terms are seen to be more accurate than the Padé [20, 20] calculations on 40 terms given by Graffi *et al* and almost as accurate as their Padé–Borel [10, 10] calculations on 20 terms. The accuracy is better than the results of Biswas *et al* for $\lambda = 0.1$ but rapidly deteriorates for larger λ. However, there is no bias evident except for $\lambda = 100$ where the effect of truncating at the first erratic term is to stop the series too early and so underestimate the sum of the series.

4. Oscillators with finite potential barriers

There are several reasons for wishing to calculate the energy of an oscillator with a finite potential barrier.

First, in classical Newtonian mechanics, nonlinear oscillators exist with both negative and positive perturbation terms. The behaviour of the oscillator depends only on the value of the potential energy up to the maximum amplitude of oscillation, and is entirely independent of any hypothetical potential energy at larger amplitudes. In quantum mechanics the cut-off is not so sharp, because of the phenomenon of leakage and diffuse wavefunctions. However, continuity with the classical case does suggest that (i) the wavefunctions will be small at infinity, (ii) the energy levels will be governed mainly by the potential near the origin and (iii) that conditions at infinity will have little effect on the energy levels.

Second, the WKB solution of the Schrödinger equation for an oscillator with mth power energy distortion, namely equation (1), is

$$\psi = \frac{A}{(E - x^2 - \lambda x^m)^{1/4}} \exp\left(\pm \int (E - x^2 - \lambda x^m)^{1/2} \, dx\right).$$

This contains an outer oscillatory part of the solution if m is odd and/or λ is negative. According to Weyl's theory for singular self-adjoint equations (Weyl 1910; see also Titchmarsh 1946) we can, by choosing a complex E and the ($\pm$) sign, find a solution which both represents an outgoing wave and is square integrable outside the potential well.

Third, in the Stark effect for the hydrogen atom, where the potential field is tilted by a constant force, the wall of the potential well is finite on one side. Silverstone (1978) obtained a perturbation series for the energy levels which was a divergent series of negative terms. For small values of the perturbation parameter the terms of the series decreased to a minimum, after which they grew steadily. When he truncated just before the minimum term, he obtained a value for the energy in close agreement with the calculations of Hehenberger *et al* (1974) using Weyl's theory.

These three considerations suggest (i) that decaying energy levels represented by complex eigenvalues exist for the anharmonic oscillator, and furthermore (ii) that they might be deducible directly from the coefficients of the asymptotic series with minimal consideration of conditions at infinity.

5. Summing a series of positive terms

The choice of a method of summing a divergent series of positive terms depends on several considerations.

First we assume that the series is asymptotic. Hence, if the terms rapidly decrease in size, the remainder after truncation is approximately equal to the first term omitted. We can also improve on this if the ratio of successive terms varies slowly along the series and is less than 1. We can then use some convergence speeding process such as the Aitken δ_2 transformation. In any case, when λ is small we sum the asymptotic series like any other convergent series. In our case, this occurs approximately for $\lambda < 0.02$ for the quartic and $\lambda < 0.15$ for the cubic distortion.

For intermediate values of λ the early terms still decrease, then level off and grow without limit. Finally, if λ is larger than approximately 0.4 in both cases, the terms grow from the start.

Many of the rational transformations have singularities within the operating range of these series, and so cannot be used or must be modified.

For instance the Borel transformation

$$\sum u_n = \int_0^\infty \sum \frac{u_n t^n}{n!} e^{-t}\, dt$$

as used successfully by Graffi *et al* (1970) on an alternating series could also be used on a series of positive terms. If the series $\Sigma\, u_n t^n/n!$ can be analytically continued to moderately large t and has a simple pole in $(0, \infty)$, then the path of integration must be taken round the singularity and will yield a complex integral. This could be the complex eigenvalue we wish to evaluate.

However, it was decided that it would be simpler to use two different models for the series for the intermediate range of λ. These are the exponential integral, which is closely related to the Borel transformation, and a truncated binomial. In Drummond (1981) these two functions are proved to be asymptotically the same. Furthermore, when the truncated binomial was tested on one of Silverstone's series for the Stark effect, it gave a complex sum agreeing with both the real and imaginary parts of Hehenberger's complex eigenvalues calculated using Weyl's theory. This is more precise then Silverstone's truncation described in § 4.

The exponential integral. If the series behaves like the exponential integral,

$$e^{-x}E_i(x) = -\int_{-x}^{\infty} \frac{e^{-x-t}}{t}\, dt = \frac{1}{x} + \frac{1}{x^2} + \ldots + \frac{(n-1)!}{x^n} + (-1)^{n+1} \int_{-x}^{\infty} \frac{n!\, e^{-x-t}}{t^{n-1}}\, dt$$

where x is close to n, $u_n = (n-1)!/x^n$ and e^{-x} is small, then the remainder integral is

$$(-1)^{n+1} \int_{-x}^{-n} \frac{n!\, e^{-x-t}}{t^{n+1}}\, dt - \int_0^{\pm\pi} \frac{n!}{n^n} \exp(-x + n\, e^{i\theta} - ni\theta) i\, d\theta + (-1)^{n+1} e^{-2x} \int_x^{\infty} \frac{n!\, e^{x-t}}{t^{n+1}}\, dt.$$

The first of these three integrals is approximately $(x-n)u_n$, the second is $[-\frac{1}{3} \pm i(n\pi/2)^{1/2}]u_n$ and the third is small. Hence the remainder near the smallest term is approximately

$$u_n[x - n - \tfrac{1}{3} \pm i(n\pi/2)^{1/2}].$$

If the sequence of terms is concave, the three smallest terms being u_{k-1}, u_k, u_{k+1}, then we can substitute for x and n, so the sum of the series is

$$S \simeq u_0 + u_1 + \ldots + u_{k-1} + \frac{(u_{k-1} - u_k)u_n}{u_{k-1} - 2u_k + u_{k+1}} - \frac{u_k}{3} + iu_k \left(\frac{\pi u_{k-1} u_k}{2(u_{k-1}u_{k+1} - u_k^2)} \right)^{1/2}. \tag{3}$$

The truncated binomial series.

Let

$$T_n = \frac{u_n r!}{p(p+1)\ldots(p+r-1)z^r} \left((1-z)^{-p} - 1 - pz - \ldots - \frac{p(p+1)\ldots(p+r+2)}{(r-1)!} z^{r-1} \right) \tag{4}$$

$$= u_n \left\{ 1 + \frac{p+r}{r+1} z \left[1 + \frac{p+r+1}{r+2} z \left(1 + \frac{p+r+2}{r+3} z(1+\ldots) \right) \right] \right\} \tag{5}$$

and

$$Z_n = u_n [\![1 + R_n \{ 1 + R_{n+1} [1 + R_{n+2}(1+\ldots)] \}]\!] \tag{6}$$

where $R_n = u_{n+1}/u_n$.

If Z_n is the tail of the series to be summed, we may match the first four items of the series (5) and (6) using the parameters p, r and z found from the equations

$$r = -(R_n - 4R_{n+1} + 3R_{n+2})/(R_n - 2R_{n+1} + R_{n+2}), \tag{7}$$

$$z = (r+2)R_{n+1} - (r+1)R_n, \tag{8}$$

$$p = -r + (r+1)R_n/z, \tag{9}$$

and hence find T_n using equation (4).

If the series (6) is the tail of the exponential integral series, all the terms of the two series coincide in the limit $r \to \infty$. Hence the limiting truncated binomial and the exponential integral agree.

For the $E_n(\lambda)$ series, r was not a positive integer, so I relaxed equation (7), chose $r = 4, 8, 16, 32$, solved equations (8), (9) and (4) using $(1-z)^{-p} = (z-1)^{-p}\, e^{ip\pi}$ if $z > 1$, and extrapolated to zero error in the fourth term to find T_n.

To estimate the error in this method this calculation was repeated for T_{n+1} and T_{n+2}. The series was then transformed by term splitting to

$$\begin{aligned} S = {} & u_0 + u_1 + \ldots + u_{n-1} + [(T_n) + (u_n - T_n)] + [(T_{n+1}) + (u_{n+1} - T_{n+1})] \\ & + [(T_{n+1}) + (u_{n+2} - T_{n+2})] \\ = {} & u_0 + u_1 + \ldots + u_{n-1} + T_n + u_n^* + u_{n+1}^* + \ldots \end{aligned}$$

where $u_n^* = u_n - T_n + T_{n+1}$. This sums the series, while the last two terms are part of a new asymptotic series which can be used to judge the accuracy of the transformation.

Both equations (3) and (4) were used near the smallest terms to sum the series for intermediate λ when a smallest term was available.

For the largest values of λ the truncated binomial T_0 together with u_0^* and u_1^* were used and the calculated eigenvalues were rounded off larger than u_0^* and u_1^*.

The results of the truncated binomial calculations are listed in tables 4, 5, 6 and 7.

The calculations using equation (3) were also checked against these and found to agree for intermediate λ where a minimum existed.

In the case of the cubic distortion, it was speculated that the appropriate model of the asymptotic series which contains only even powers of λ might be a truncated double binomial based on $[(1+z)^p + (1-z)^p]$. However, the properties of this series are not fully developed in the first 20 terms of the cubic distortion series, and the single binomial series was found to fit better to these terms. Also the differences between the two calculations were less than the size of the terms in the transformed series.

6. Conclusion

These calculations using perturbation series have not been rigorously justified. However, if the series are asymptotic then, for very small λ, the error is comparable with the first term omitted and very accurate calculations may be made. For intermediate λ, the three models (exponential integral, truncated binomial and truncated double binomial) mutually agree and represent a significant improvement on truncation before the smallest term.

For large λ, where the first term is the smallest term, the calculations are self consistent.

Table 4. Real part of the energy levels for the oscillator with quartic distortion for negative λ calculated by fitting a truncated binomial to the four smallest terms of the series.

$-\lambda$	E_0	E_1	E_2	E_3	E_4
0	1	3	5	7	9
0.01	0.992 363 220 6	2.961 401 903 5	4.898 302 036 6	6.801 432 758 5	8.668 928 127 8
0.02	0.984 427 669 8	2.920 282 161 3	4.786 335 05	6.573 552	8.268 8
0.03	0.976 146 197 4	2.875 948 30	4.659 247	6.291 08	7.698
0.04	0.967 451 234	2.827 103	4.504 5	5.91	7.05
0.05	0.958 233 36	2.771 26	4.315	5.54	6.6
0.06	0.948 330	2.707	4.12	5.3	6.4
0.07	0.937 582	2.636	3.96	5.1	6.3
0.08	0.925 95	2.566	3.83	5.0	6.3
0.09	0.913 55	2.50	3.7	4.9	6.2
0.1	0.900 6	2.45	3.7	4.9	6.2
0.12	0.874 6	2.35	3.6	4.8	6.2
0.15	0.839	2.26	3.5	4.8	6.3
0.2	0.793	2.18	3.5	4.8	7
0.25	0.76	2.14	3.5	5	7
0.3	0.74	2.1	3.6	5	7
0.4	0.72	2.1	4	5	7
0.5	0.72	2.2	4	6	8
0.6	0.72	2.3	4	6	8
0.7	0.72	2.3	4	6	8
0.8	0.73	2.3	4	6	9
0.9	0.74	2.4	4	6	9
1	0.75	2	4	7	9

Table 5. Imaginary part of the energy levels in table 4.

$-\lambda$	E_0	E_1	E_2	E_3	E_4
0					
0.01					
0.02					0.000 04
0.03			0.000 054	0.002 35	0.042 6
0.04		0.000 089	0.004 60	0.076	0.41
0.05	0.000 014 6	0.001 54	0.042	0.31	0.9
0.06	0.000 119	0.008 9	0.13	0.62	1.5
0.07	0.000 521	0.027	0.27	0.9	2.0
0.08	0.001 54	0.057	0.41	1.2	2.4
0.09	0.003 49	0.10	0.56	1.5	2.8
0.1	0.006 6	0.15	0.7	1.7	3.1
0.12	0.016 5	0.25	1.0	2.2	3.7
0.15	0.039	0.4	1.4	2.8	4.5
0.2	0.09	0.7	1.9	3.6	5.6
0.25	0.14	0.9	2.4	4.2	6
0.3	0.19	1.1	2.7	4.8	7
0.4	0.27	1.4	3.4	5.8	8
0.5	0.35	1.7	3.8	6	9
0.6	0.42	1.9	4	7	10
0.7	0.48	2.1	5	8	11
0.8	0.53	2.3	5	8	12
0.9	0.58	2.5	5	9	13
1	0.62	2.6	5	9	13

Table 6. Real part of the energy levels for the oscillator with cubic distortion calculated by fitting a truncated binomial to the four smallest terms of the series.

λ	E_0	E_1	E_2	E_3	E_4	E_5
0	1	3	5	7	9	11
0.01	0.999 931 231 826	2.999 556 030 071	4.998 805 331 74	6.997 678 805 0	8.996 176 117 35	10.994 296 935 57
0.02	0.999 724 708 753	2.998 221 471 381	4.995 210 246 59	6.990 685 658 8	8.984 642 287 57	10.977 074 667 08
0.03	0.999 379 771 595	2.995 988 303 218	4.989 181 036 35	6.978 930 181 4	8.965 207 416 03	10.947 983 869 60
0.04	0.998 895 309 802	2.992 842 905 840	4.980 659 915 30	6.962 255 845 9	8.937 537 029 9	10.906 406 445 4
0.05	0.998 269 743 044	2.988 765 663 646	4.969 562 432 1	6.940 430 237 1	8.901 126 157 5	10.851 392 942 9
0.06	0.997 500 994 288	2.983 730 368 840	4.955 773 448 0	6.913 129 345 9	8.855 253 866 6	10.781 553 284 1
0.07	0.996 586 453 093	2.977 703 377 7	4.939 141 178 9	6.879 913 082 3	8.798 908 377	10.694 865 910
0.08	0.995 522 927 280	2.970 642 444 9	4.919 468 460 3	6.840 186 864	8.730 660 68	10.588 327 5
0.09	0.994 307 658 0	2.962 495 123 0	4.896 499 81	6.793 139 44	8.648 437	10.457 247
0.1	0.992 932 851	2.953 196 547 3	4.869 901 7	6.737 636 6	8.549 073	10.293 58
0.12	0.989 690 721	2.930 803 972	4.803 884	6.593 70	8.272 3	9.76
0.15	0.983 476 9	2.885 742	4.658	6.208	7.50	8.9
0.2	0.968 63	2.756 5	4.19	5.6	7.15	8.8
0.25	0.944 8	2.55	4.0	5.6	7.38	9.3
0.3	0.910	2.43	4.0	5.8	7.8	9.8
0.4	0.85	2.43	4.3	6.4	8.6	10.9
0.5	0.82	2.6	4.7	7	9	12
0.6	0.83	2.7	5	8	10	13
0.7	0.85	2.9	5	8	11	14
0.8	0.9	3	6	8	11	14
0.9	0.9	3	6	9	12	15
1	1	3	6	9	12	15

Table 7. Imaginary part of the energy levels in table 6.

λ	E_0	E_1	E_2	E_3	E_4	E_5
0.1						
0.12						0.022
0.15			0.000 2	0.03	0.34	0.98
0.2	0.000 02	0.006	0.20	0.8	1.7	2.7
0.25	0.001 4	0.12	0.8	1.7	2.8	4.0
0.3	0.013	0.35	1.3	2.4	3.6	5.0
0.4	0.08	0.8	2.0	3.4	4.8	6.4
0.5	0.18	1.2	2.6	4	6	7.4
0.6	0.27	1.4	3	5	6	8
0.7	0.34	1.7	3	5	7	9
0.8	0.4	2	4	5	7	10
0.9	0.5	2	4	6	8	10
1	0.5	2	4	6	8	11

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